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New exact solutions for randomly loaded beams with stochastic flexibility

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Abstract

Presently there exist several hundred papers on so-called 'stochastic finite element technique' but extremely few closed-form solutions are available for meaningful comparison. This paper intends to fill this huge gap. This study deals with deformation of deterministic beams or stochastic beams subjected to random excitation. Exact solutions are formulated for four different classes of problems. These solutions can serve as benchmark solutions to be utilized for assessing the performance of various approximate, analytical or numerical techniques. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

Exact solutions for the bending of deterministic beams under random loads were treated by Lomakin (1996), Rzhanitsyn (1977, 1978) and Elishakoff (1983). Exact solutions for beams with stochastic stiffness were studied by Köylüoğlu et al. (1994) and Elishakoff et al. (1995). Elishakoff et al. (1995) treated stochastic beams under deterministic excitation with arbitrary covariance function of the stochastic flexibility, whereas Köylüoğlu et al. (1994) considered a case of both loading and stiffness being stochastic. The latter paper used the concept of the Green's function and the spatial spectral densities, whereas the former contained a correlation analysis.

Formulation of Köylüoğlu et al. is convenient when the random flexibility is given in terms of the spatial spectral density which is identically vanishing outside some region of spatial frequencies. In other cases, the correlation analysis performed by Elishakoff et al. (1995) appears to be advantageous.

Here we present derivation for the probabilistic characteristics of the beam's bending moment

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and displacement through first establishing stochastic differential equations, governing the beam's displacement. The treatment necessitates knowledge of characteristics of deterministic or random flexibility throughout analysis, rather than those of stiffness. Following special cases are considered: (a) deterministic beam under random loading represented as a random field; (b) stochastic beam with flexibility depending on a random variable subjected to a load treated as a random field; (c) beam with random flexibility represented as a random field subjected to a random load depending on a random variable, and, finally, (d) a stochastic beam subjected to a load with both flexibility and load treated as random fields. It is shown that the previous analyses by Rzhanitsyn (1978) and Elishakoff et al. (1955) are particular cases of the present formulation. Most importantly, the nontrivial examples presented could be used for meaningful comparison of the FEM's stochastic version with closed-form solutions developed in the present paper.

2. Deterministic beam under random load represented as a random field

The field of bending moment $m(x)$ of a statically determined Beurnoulli–Euler beam of length L, subject to distributed load $q(x)$ satisfies the following differential equation:

$$\frac{\mathrm{d}^2 m(x)}{\mathrm{d}x^2} = q(x) \tag{1}$$

Assume that the load $q(x)$ is a random field with a given mean and covariance function. By applying expectation operation to eqn (1), we obtain

$$\frac{\mathrm{d}^2 \bar{m}(x)}{\mathrm{d}x^2} = \bar{q}(x) \tag{2}$$

where $\bar{m}(x) = E[m(x)]$ is the mean function of the bending moment $m(x)$, and $\bar{q}(x) = E[q]$ is the mean function of the load. Subtracting eqn (2) from eqn (1) and multiplying the resulting equation by its counterpart that is evaluated at the cross-section y, we obtain

$$\frac{\mathrm{d}^2[m(x)-\bar{m}(x)]}{\mathrm{d}x^2}\frac{\mathrm{d}^2[m(y)-\bar{m}(y)]}{\mathrm{d}y^2} = [q(x)-\bar{q}(x)]\,[q(y)-\bar{q}(y)] \tag{3}$$

Applying expectation operation to eqn (3), we arrive at the governing equation for the covariance function of bending moment

$$\frac{\partial^4 C_m(x,y)}{\partial x^2\,\partial y^2} = C_q(x,y) \tag{4}$$

where $C_m(x,y)$ and $C_q(x,y)$ are, respectively, covariance function of bending moment and load:

$$C_m(x,y) = E\{[m(x)-\bar{m}(x)]\,[m(y)-\bar{m}(y)]\} \tag{5}$$

$$C_q(x,y) = E\{[q(x)-\bar{q}(x)]\,[q(y)-\bar{q}(y)]\} \tag{6}$$

Solutions to eqn (4) are composed of a complementary solution $\psi(x,y)$ and a particular solution $\phi(x,y)$. The complementary solution can be written as follows

$$\psi(x,y) = f_1(x) + yf_2(x) + g_1(y) + xg_2(y) \tag{7}$$

where $f_1(x)$, $f_2(x)$, $g_1(y)$ and $g_2(y)$ are four arbitrary functions of their respective arguments. Thus

$$C_m(x,y) = f_1(x) + yf_2(x) + g_1(y) + xg_2(y) + \phi(x,y) \tag{8}$$

Note that the particular solution depends on the form of the load's correlation function, and is found by the quadruple integration of the right hand side of eqn (4). Let us consider the special case of the load's correlation function that depends upon $|x-y|$. This type of correlation function is often encountered in engineering practice. In this case the integration domain should be split into two parts: one in which $x \geqslant y$ and the other with $x < y$. We first integrate the eqn (4) twice with respect to x and y:

$$\phi''(u,v) = \int_0^v \int_0^y C_q(x,y)\,\mathrm{d}y\,\mathrm{d}x + \int_0^v \int_y^u C_q(x,y)\,\mathrm{d}y\,\mathrm{d}x; \quad \text{for } u \geqslant v \tag{9}$$

The final form of the particular solution reads:

$$\phi(x,y) = \int_0^y \int_0^v \phi''(v,u)\,\mathrm{d}v\,\mathrm{d}u + \int_0^y \int_v^x \phi''(u,v)\,\mathrm{d}v\,\mathrm{d}u; \quad \text{for } x \geqslant y \tag{10}$$

For specificity, here and hereinafter, let us consider a beam that is simply supported at both its ends. For a beam that is clamped at $x = 0$ and free at $x = L$ the pertinent derivations are given in the Appendix.

The boundary conditions for the bending moment $m(x)$ are:

$$m(0) = 0; \quad m(L) = 0 \tag{11}$$

By taking the expectation operator, we have

$$\bar{m}(0) = 0; \quad \bar{m}(L) = 0 \tag{12}$$

For the arbitrary bending moment $m(y)$ we obtain:

$$\begin{aligned} C_m(0,y) &= E\{[m(0)-\bar{m}(0)]\,[m(y)-\bar{m}(y)]\} = 0 \\ C_m(L,y) &= E\{[m(L)-\bar{m}(L)]\,[m(y)-\bar{m}(y)]\} = 0 \end{aligned} \tag{13}$$

and similarly

$$C_m(x,0) = 0; \quad C_m(x,L) = 0 \tag{14}$$

The boundary conditions (13) and (14) for eqn (4) in explicit form read:

$$\begin{aligned} &f_1(0) + yf_2(0) + g_1(y) + \phi(y,0) = 0; \quad f_1(L) + yf_2(L) + g_1(y) + Lg_2(y) + \phi(L,y) = 0 \\ &f_1(x) + g_1(0) + xg_2(0) + \phi(x,0) = 0; \quad f_1(x) + Lf_2(x) + g_1(L) + xg_2(L) + \phi(L,x) = 0 \end{aligned} \tag{15}$$

By solving for $f_1(x)$, $f_2(x)$, $g_1(y)$ and $g_2(y)$ and noting that $\phi(x,0) = \phi(y,0)$, the covariance function becomes

$$C_m(x,y) = \phi(x,y) + \frac{xy}{L^2}\phi(L,L) - \frac{1}{L}[x\phi(L,y) + y\phi(L,x)], \quad \text{for } x \geqslant y \tag{16}$$

$$C_m(x,y) = \phi(y,x) + \frac{xy}{L^2}\phi(L,L) - \frac{1}{L}[x\phi(L,y) + y\phi(L,x)], \quad \text{for } x \leqslant y \tag{17}$$

Note that an analogous general form for different case was obtained by Elishakoff et al. (1995).

Once the expressions for the mean bending moment $\bar{m}(x)$ and its covariance function $C_m(x,y)$ are obtained a similar technique should be resorted to in order to derive the mean displacement $\bar{w}(x)$ and the displacement's covariance function $C_w(x,y)$. Indeed, the beam's displacement $w(x)$ satisfies the following relations:

$$\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} = f_0(x)m(x) \tag{18}$$

$$\frac{\mathrm{d}^2 \bar{w}(x)}{\mathrm{d}x^2} = f_0(x)\bar{m}(x) \tag{19}$$

$$\begin{aligned}\frac{\partial^4 C_w(x,y)}{\partial x^2\,\partial y^2} &= E\{[f_0(x)m(x) - f_0(x)\bar{m}(x)]\,[f_0(y)m(y) - f_0(y)\bar{m}(y)]\}\\ &= f_0(x)f_0(y)C_m(x,y)\end{aligned} \tag{20}$$

where the beam's flexibility

$$f_0(x) = \frac{1}{EI(x)} \tag{21}$$

is a deterministic function.

Note that, for a simply supported beam loaded with distributed loading $q(x)$, eqn (20) and eqn (4) have the same type of boundary conditions. It follows that a similar procedure to that used to determine the covariance function of bending moment can be resorted to in order to obtain the covariance function of displacement. Thus,

$$C_w(x,y) = \chi(x,y) + \frac{xy}{L^2}\chi(L,L) - \frac{1}{L}[x\chi(L,y) + y\chi(L,x)], \quad \text{for } x \geqslant y \tag{22}$$

$$C_w(x,y) = \chi(y,x) + \frac{xy}{L^2}\chi(L,L) - \frac{1}{L}[x\chi(L,y) + y\chi(L,x)], \quad \text{for } x \leqslant y \tag{23}$$

where $\chi(x,y)$ is the particular solutions of eqn (20).

2.1. Applications

Consider a simply supported beam with deterministic flexibility when

$$\bar{q}(x) = E[q(x)] = 0; \quad C_q(x,y) = a^2 \exp\left(-\alpha\frac{|x-y|}{L}\right); \quad f_0(x) = f_0 \tag{24}$$

We introduce nondimensional axial coordinates: $\xi = x/L$ and $\eta = y/L$; for $\xi \geqslant \eta$ the particular solution of eqn (4) reads

$$\phi(\xi,\eta) = a^2 L^4 \left(\frac{3\xi\eta^2-\eta^3}{3\alpha} - \frac{\xi\eta}{\alpha^2} + \frac{\xi-\xi\,e^{-\alpha\eta}-\eta-\eta\,e^{-\alpha\xi}}{\alpha^3} + \frac{1-e^{-\alpha\xi}-e^{-\alpha\eta}+e^{\alpha(-\xi+\eta)}}{\alpha^4}\right) \tag{25}$$

Here and hereinafter, the particular solution for $\xi \leqslant \eta$ can be obtained from eqn (10) by formal replacement of ξ by η and η by ξ, owing to symmetry in ξ and η. Substituting eqn (25) in eqn (16) the following expressions for the covariance and variance of the bending moment are obtained for $\xi \geqslant \eta$

$$\begin{aligned} C_m(\xi,\eta) = a^2L^4 \Bigg[&\frac{1}{3\alpha}(2\xi\eta-3\xi^2\eta+\xi^3\eta-\eta^3+\xi\eta^3) + \frac{1}{2\alpha^3}(-\eta+\xi\eta) \\ &+ \frac{1}{\alpha^4}(1-e^{-\alpha\xi}-e^{-\alpha\eta}+e^{\alpha(-\xi+\eta)}-\xi+\xi\,e^{-\alpha}-e^{\alpha(-1+\eta)}\xi \\ &+\xi\,e^{-\alpha\eta}-\eta+\eta\,e^{-\alpha}-e^{\alpha(-1+\xi)}\eta+\eta\,e^{-\alpha\xi}+2\xi\eta-2\xi\eta\,e^{-\alpha}) \Bigg] \end{aligned} \tag{26}$$

$$\begin{aligned} C_m(\xi,\xi) = a^2L^4 \Bigg[&\frac{1}{3\alpha}(2\xi^2-4\xi^3+2\xi^4) + \frac{1}{2\alpha^3}(-\xi+\xi^2) + \frac{1}{\alpha^4}(2-2\,e^{-\alpha\xi} \\ &-2\xi+2\xi\,e^{-\alpha}-2e^{\alpha(-1+\xi)}\xi+2\xi\,e^{-\alpha\xi}+2\xi^2-2\xi^2\,e^{-\alpha}) \Bigg] \end{aligned} \tag{27}$$

The variance of the bending moment $C_m(\xi,\xi)$ coincides with the expression obtained by a different scheme by Rzhanitsyn (1978) that is valid for a particular form of the excitation.

Thus, the present formulation contains as a particular case Rzhanitsyn's solution. In the present formulation the form of the load's mean function and load's covariance functions is arbitrary. For displacement's variance $C_w(\xi,\xi)$, from eqn (20) we get, for $\alpha = 1$

$$\begin{aligned} C_w(\xi,\xi) = f_0^2 a^2 L^8 \Bigg[&2-2\,e^{-\xi}+\frac{\xi}{3}(-14+5\,e^{-1}-5\,e^{-1+\xi}+8\,e^{-\xi}) \\ &+\xi^2\left(\frac{23{,}053}{3780}-\frac{20\,e^{-1}}{9}-e^{-\xi}\right)+\xi^3\left(-3+\frac{7\,e^{-1}}{6}-\frac{e^{-1+\xi}}{3}+\frac{e^{-\xi}}{3}\right) \\ &+\xi^4\left(\frac{997}{540}-\frac{13\,e^{-1}}{18}\right)+\xi^5\left(-\frac{13}{30}+\frac{e^{-1}}{6}\right)+\xi^6\left(\frac{47}{270}-\frac{e^{-1}}{18}\right)-\frac{8\xi^7}{315}+\frac{2\xi^8}{315} \Bigg] \end{aligned} \tag{28}$$

Figures 1 and 2 portray the covariance and variance functions, respectively, for the bending moment and the deflection; the value of α was fixed at unity. Both covariance functions reach their maxima at $x = y = L/2$, due to the homogeneity of the random fields chosen.

3. Beam with random flexibility represented by a random variable, and subjected to a random field load

In this case it is instructive to replace the flexibility by the related nondimensional random variable α_f, namely

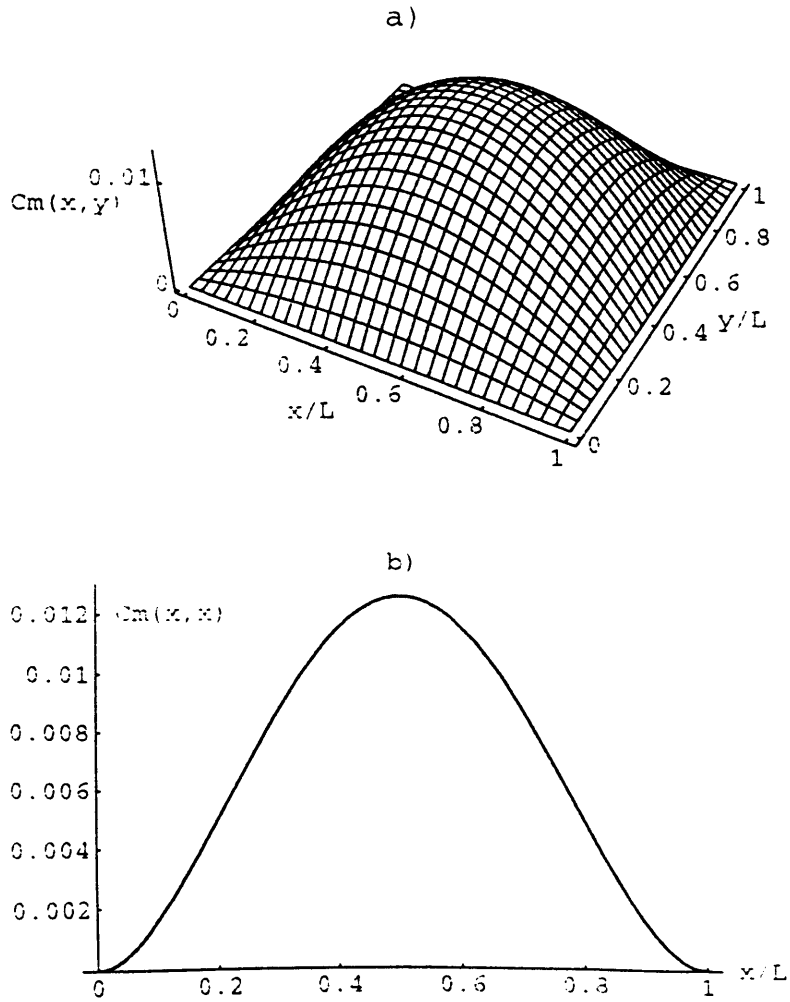

Fig. 1. Probabilistic characteristics of the bending moment for a deterministic beam subjected to a random load represented as a random field, normalized by a^2L^4: (a) covariance function; (b) variance function.

$$f(x) = f_0(x)[1+\alpha_f] \tag{29}$$

where $f_0(x)$ is a deterministic function and α_f is a random variable with zero mean. The mean and the covariance function for the flexibility are, respectively,

$$\bar{f}(x) = f_0(x) \tag{30}$$

$$C_f(x,y) = f_0(x)f_0(y)\,\mathrm{Var}\,[\alpha_f] \tag{31}$$

The problem is governed by the following equations, for uncorrelated $q(x)$ and $f(x)$:

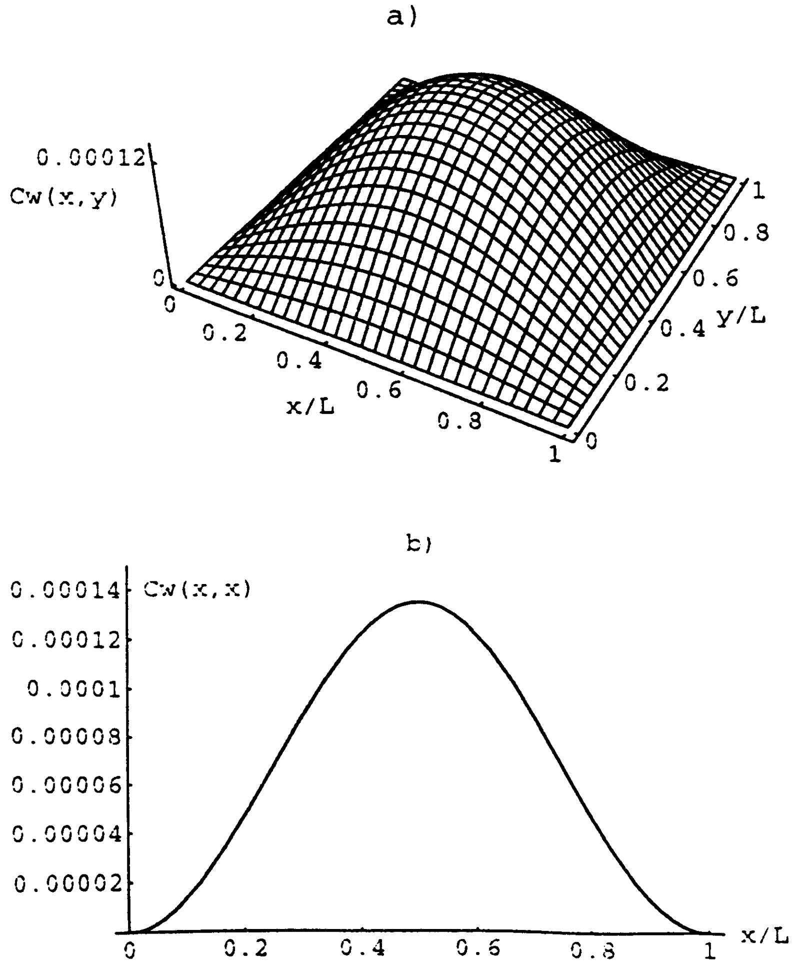

Fig. 2. Probabilistic characteristics of the displacement for a deterministic beam subjected to a random load represented as a random field, normalized by $f_0^2 a^2 L_8$: (a) covariance function; (b) variance function.

$$\frac{\mathrm{d}^2 \bar{w}(x)}{\mathrm{d}x^2} = \bar{m}(x) f_0(x) \tag{32}$$

$$\begin{aligned}\frac{\partial^4 C_w(x,y)}{\partial x^2\,\partial y^2} &= E\{[m(x)f(x) - \bar{m}(x)f_0(x)]\,[m(y)f(y) - \bar{m}(y)f_0(y)]\} \\ &= E[m(x)m(y)]E[f(x)f(y)] - \bar{m}(x)\bar{m}(y)f_0(x)f_0(y) \\ &= [C_m(x,y) + \bar{m}(x)\bar{m}(y)]f_0(x)f_0(y)(\mathrm{Var}\,[\alpha_f]+1) - \bar{m}(x)\bar{m}(y)f_0(x)f_0(y) \\ &= f_0(x)f_0(y)[C_m(x,y)(\mathrm{Var}\,[\alpha_f]+1) + \bar{m}(x)\bar{m}(y)\,\mathrm{Var}\,[\alpha_f]\,]\end{aligned} \tag{33}$$

where $\bar{m}(x)$ and $C_m(x,y)$ can be obtained by eqn (2) and (4), since the beam is statically determinate.

3.1. Applications

Consider the following characteristics for load and flexibility

$$\bar{q}(x) = 0; \quad C_q(x,y) = a^2\left(1+\frac{|x-y|}{L}\right)\exp\left(-\frac{|x-y|}{L}\right); \quad f_0(x) = f_0 \tag{34}$$

In this case the particular solution of eqn (33), for $\xi \geqslant \eta$, reads

$$\chi(\xi,\eta) = f_0^2 a^2 L^8 (\mathrm{Var}\,[\alpha_f]+1)(\chi_1(\xi,\eta)+\chi_2(\xi,\eta)) \tag{35}$$

The functions χ_1 and χ_2 are not reproduced here.

We obtain the following expression for the variance of the displacement

$$C_w(\xi,\xi) = f_0^2 a^2 L^8 (\mathrm{Var}\,[\alpha_f]+1)\Bigg[18-18\,e^{-\xi}+\frac{\xi}{3}(-116+52\,e^{-1}-52\,e^{-1+\xi}+62\,e^{-\xi})+\xi^2\left(\frac{36{,}181}{756}-22\,e^{-1}+\frac{5\,e^{-1+\xi}}{3}-\frac{13\,e^{-\xi}}{3}\right)+\xi^3\left(-19+\frac{29\,e^{-1}}{3}-\frac{8\,e^{-1+\xi}}{3}+\frac{4\,e^{-\xi}}{3}\right)+\xi^4\left(\frac{5969}{540}-\frac{17\,e^{-1}}{3}+\frac{e^{-1+\xi}}{3}+\frac{e^{-\xi}}{3}\right)-\xi^5\left(-\frac{19}{10}+e^{-1}\right)+\xi^6\left(\frac{187}{270}-\frac{e^{-1}}{3}\right)-\frac{16\xi^7}{315}+\frac{4\xi^8}{315}\Bigg] \tag{36}$$

Figure 3 shows the normalized variance and covariance functions for the transverse displacement for a simply supported beam in the case $\mathrm{Var}\,[\alpha_f] = 1$.

4. Beam with random field flexibility, and subjected to a loading depending on a random variable

Let the general expression for the load $q(x)$ be

$$q(x) = \alpha_q P(x) \tag{37}$$

where $P(x)$ is a deterministic function and α_q is a random variable. The mean load equals

$$\bar{q}(x) = E[\alpha_q]P(x); \quad C_q(x,y) = P(x)P(y)\,\mathrm{Var}\,[\alpha_q] \tag{38}$$

Let the general form for the flexibility be

$$f(x)+f_0(x)[1+\alpha_f(x)] \tag{39}$$

Note that eqn (39) is analogous to eqn (29) except that in new circumstances $\alpha_f(x)$ is a random field with zero mean. Hence

$$\bar{f}(x) = f_0(x); \quad C_f(x,y) = f_0(x)f_0(y)C_{\alpha_f}(x,y) \tag{40}$$

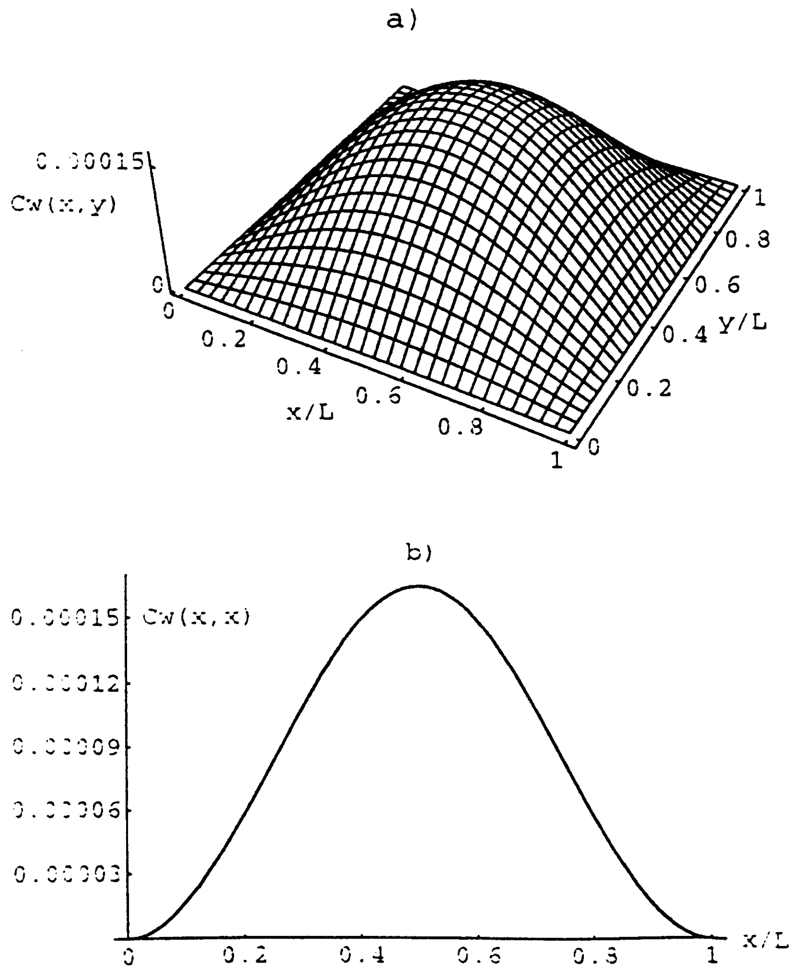

Fig. 3. Probabilistic characteristics of the displacement for a beam with random flexibility represented by a random variable and subjected to a random load represented as a random field, normalized by $f_0^2 a^2 L_8(\mathrm{Var}\,[\alpha_f]+1)$: (a) covariance function; (b) variance function.

Since in this case $C_q(x, y)$ is a separable function of x and y, eqn (4) can be solved straightforwardly. We obtain

$$\bar{m}(x) = E[\alpha_q]M(x) \tag{41}$$

$$C_m(x, y) = M(x)M(y)\,\mathrm{Var}\,[\alpha_q] \tag{42}$$

where

$$M(x) = -\int_0^x \int_0^v P(u)\,\mathrm{d}u\,\mathrm{d}v + Q_0 x + M_0 \tag{43}$$

represents the bending moment of a beam loaded with $P(x)$; M_0 and Q_0 are, respectively, the bending moment and shear force at the end $x = 0$ of the beam.

The governing equations for $\bar{w}(x)$ and $C_w(x, y)$ for uncorrelated $q(x)$ and $f(x)$ become

$$\frac{d^2\bar{w}(x)}{dx^2} = E[\alpha_q]M(x)f_0(x) \tag{44}$$

$$\begin{aligned}\frac{\partial^4 C_w(x,y)}{\partial^2 x^2\,\partial y^2} &= E\{[f(x)m(x) - f_0(x)E[\alpha_q]M(x)][f(y)m(y) - f_0(y)E[\alpha_q]M(y)]\} \\ &= E[f(x)f(y)]E[m(x)m(y)] - f_0(x)f_0(y)M(x)M(y)E[\alpha_q]^2 \\ &= f_0(x)f_0(y)(C_{\alpha_f}(x,y)+1)M(x)M(y)(\mathrm{Var}\,[\alpha_q] + E[\alpha_q]^2) \\ &\quad - f_0(x)f_0(y)M(x)M(y)E[\alpha_q]^2 \\ &= M(x)M(y)f_0(x)f_0(y)[\mathrm{Var}\,[\alpha_q](C_{\alpha_f}(x,y)+1) + E[\alpha_q]^2 C_{\alpha_f}(x,y)]\end{aligned} \tag{45}$$

4.1. Applications

Consider the following characteristics for load and flexibility

$$P(x) = P_0; \quad E[\alpha_q] = 0; \quad f_0(x) = f_0; \quad C_{\alpha_f}(x,y) = \exp\left(-\frac{|x-y|}{L}\right) \tag{46}$$

In this case the particular solution of eqn (45), for $\xi \geqslant \eta$, reads

$$\begin{aligned}\chi(\beta,\eta) = f_0^2 P_0^2\,\mathrm{Var}\,[\alpha_q]\Bigg[& 8 - 8\,e^{-\xi} - 8\,e^{-\eta} + 8\,e^{-\xi+\eta} - 2\eta - 3\eta\,e^{-\xi} - 6\eta\,e^{-\eta} - 5\,e^{-\xi+\eta}\eta \\ & - 2\eta^2\,e^{-\eta} + e^{-\xi+\eta}\eta^2 - \frac{\eta^3}{6} + \frac{\eta^4}{3} - \frac{7\eta^5}{40} + \frac{\eta^6}{30} - \frac{\eta^7}{84} + \xi(3 - 6\,e^{-\xi} - 3\,e^{-\eta} + 6\,e^{-\xi+\eta} \\ & - \frac{3\eta}{4} - \frac{9\eta\,e^{-\xi}}{4} - \frac{9\eta\,e^{-\eta}}{4} - \frac{15\,e^{-\xi+\eta}}{4} - \frac{3\eta^2\,e^{-\eta}}{4} + \frac{3\,e^{-\xi+\eta}\eta^2}{4} - \frac{\eta^3}{6} + \frac{\eta^4}{8} - \frac{\eta^5}{20} + \frac{\eta^6}{60}\Bigg) \\ & + \xi^2\left(-2\,e^{-\xi} + 2\,e^{-\xi+\eta} - \frac{3\eta\,e^{-\xi}}{4} - \frac{5\,e^{-\xi+\eta}\eta}{4} + \frac{e^{-\xi+\eta}\eta^2}{4}\right) \\ & + \xi^3\left(\frac{\eta^3}{144} - \frac{\eta^4}{288}\right) + \xi^4\left(\frac{-\eta^3}{288} + \frac{\eta^4}{576}\right)\Bigg]\end{aligned} \tag{47}$$

Substituting in eqn (22) we obtain the following expression for the variance of the displacement

$$C_w(\xi,\xi) = f_0^2 P_0^2 \,\mathrm{Var}\,[\alpha_q]\Bigg[16 - 16\,\mathrm{e}^{-\xi} + \xi(-20 + 32\,\mathrm{e}^{-1} - 32\,\mathrm{e}^{-1+\xi} + 4\,\mathrm{e}^{-\xi})$$

$$+\xi^2\left(\frac{407{,}027}{20{,}160} - 32\,\mathrm{e}^{-1} + 20\,\mathrm{e}^{-1+\xi} + 8\,\mathrm{e}^{-\xi}\right) + \xi^3\left(-\frac{2}{3} - 4\,\mathrm{e}^{-1+\xi} + 4\,\mathrm{e}^{-\xi}\right)$$

$$+\frac{155\xi^4}{144} - \frac{1387\xi^5}{1440} + \frac{317\xi^6}{720} - \frac{103\xi^7}{1008} + \frac{103\xi^8}{4032}\Bigg] \tag{48}$$

Figure 4 shows $C_w(x,y)$ and $C_w(x,x)$ in the case $\mathrm{Var}\,[\alpha_q] = 1$.

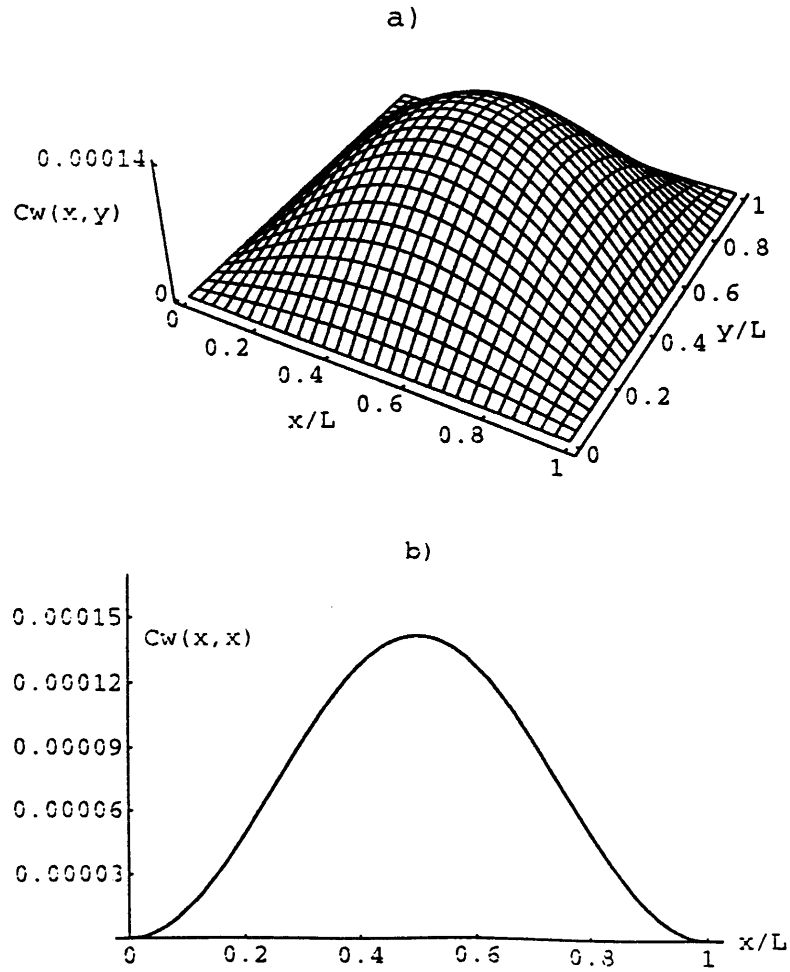

Fig. 4. Probabilistic characteristics of the displacement for a beam with random flexibility and subjected to a load depending on a random variable, normalized by $f_0^2P_0^2L_8$: (a) covariance function; (b) variance function.

5. Beam with uncorrelated random flexibility and random load

Let us consider now the case when both the flexibility $f(x)$ and the load $q(x)$ are represented as random fields. Mean and covariance functions for the moment again satisfy eqns (2) and (4). Their counterparts for the displacement satisfy the following differential equation

$$\frac{d^2\bar{w}(x)}{dx^2} = \bar{m}(x)\bar{f}(x) \tag{49}$$

$$\begin{aligned}\frac{\partial^4 C_w(x,y)}{\partial x^2\,\partial y^2} &= E\{[m(x)f(x)-\bar{m}(x)\bar{f}(x)]\,[m(y)f(y)-\bar{m}(y)\bar{f}(y)]\}\\ &= E[m(x)m(y)]E[f(x)f(y)]-\bar{m}(x)\bar{m}(y)\bar{f}\bar{f}(x)\bar{f}(y)\\ &= [C_m(x,y)+\bar{m}(x)\bar{m}(y)]\,[C_f(x,y)+\bar{f}(x)\bar{f}(y)]-\bar{m}(x)\bar{m}(y)\bar{f}(x)\bar{f}(y)\\ &= C_m(x,y)[C_f(x,y)+\bar{f}(x)\bar{f}(y)]+C_f(x,y)\bar{m}(x)\bar{m}(y)\end{aligned} \tag{50}$$

5.1. Applications

Consider the flexibility given in eqns (39)–(40). Then eqns (49)–(50) take the following forms, respectively,

$$\frac{d^2\bar{w}(x)}{dx^2} = \bar{m}(x)f_0(x) \tag{51}$$

$$\frac{\partial^4 C_w(x,y)}{\partial x^2\,\partial y^2} = f_0(x)f_0(y)[C_m(x,y)(C_{\alpha_f}(x,y)+1)+C_{\alpha_f}(x,y)\bar{m}(x)\bar{m}(y)] \tag{52}$$

Consider the following characteristics for load and flexibility

$$E[q(x)] = 0;\quad C_q(x,y) = a^2\left(1-\frac{|x-y|}{L}\right);\quad f_0(x) = f_0;\quad C_{\alpha_f}(x,y) = \exp\left(-\frac{|x-y|}{L}\right) \tag{53}$$

In this case the particular solution of eqn (50), for $\xi \geqslant \eta$ reads

$$\chi(\xi,\eta) = f_0^2a^2L^8(\chi_1(\xi,\eta)+\xi\chi_2(\xi,\eta)+\xi^3(\xi,\eta)) \tag{54}$$

The functions χ_1, χ_2 and χ_3 are not reproduced here.

We obtain the following expression for the variance of the displacement

$$C_w(\xi,\xi) = f_0^2a^2L^8\left[\frac{248}{15}-\frac{248\,e^{-\xi}}{15}+\xi\left(-\frac{197}{15}+\frac{186\,e^{-1}}{5}-\frac{186\,e^{-1+\xi}}{5}-\frac{17\,e^{-\xi}}{5}\right)\right.$$
$$\left.+\xi^2\left(\frac{458{,}249}{129{,}600}-\frac{186\,e^{-\xi}}{5}+\frac{344\,e^{-1+\xi}}{15}+\frac{304\,e^{-\xi}}{15}\right)+\xi^3\left(\frac{1819}{90}-\frac{13\,e^{-1+\xi}}{3}-e^{-\xi}\right)\right.$$

$$+\xi^4\left(-\frac{63{,}893}{5040}+\frac{8\,\mathrm{e}^{-1+\xi}}{3}+\frac{\mathrm{e}^{-\xi}}{3}\right)+\xi^5\left(\frac{8573}{2700}-\frac{2\,\mathrm{e}^{-1+\xi}}{3}+\frac{4\,\mathrm{e}^{-\xi}}{15}\right)$$

$$+\xi^6\left(-\frac{26{,}203}{21{,}600}+\frac{\mathrm{e}^{-1+\xi}}{15}+\frac{\mathrm{e}^{-\xi}}{15}\right)+\frac{1093\xi^7}{7560}-\frac{3091\xi^8}{60{,}480}+\frac{181\xi^9}{18{,}144}-\left.\frac{181\xi^{10}}{90{,}720}\right] \quad (55)$$

Figure 5 portrays the exact solution for this case. Note that if the beam's flexibility is deterministic we have to substitute $C_{\alpha f}(x, y) = 0$. Then eqn (52) reduces to eqn (20). In the case of the flexibility depending on a random variable, i.e. $\alpha_f(x) = \alpha_f$, the eqn (52) reduces to eqn (33). Finally, if flexibility is represented as a random field and the load depends on a random variable α_q, as in eqn (37), then

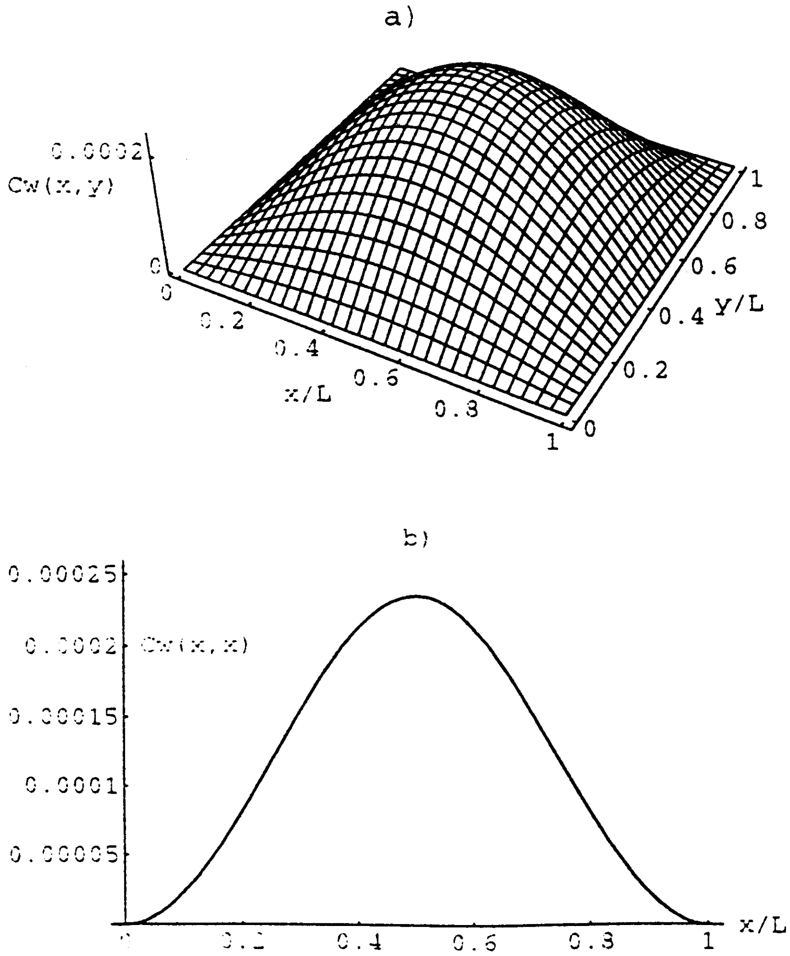

Fig. 5. Probabilistic characteristics of the displacement for a beam with uncorrelated random flexibility and random load, normalized by $f_0^2 a_0^2 L_8$: (a) covariance function; (b) variance function.

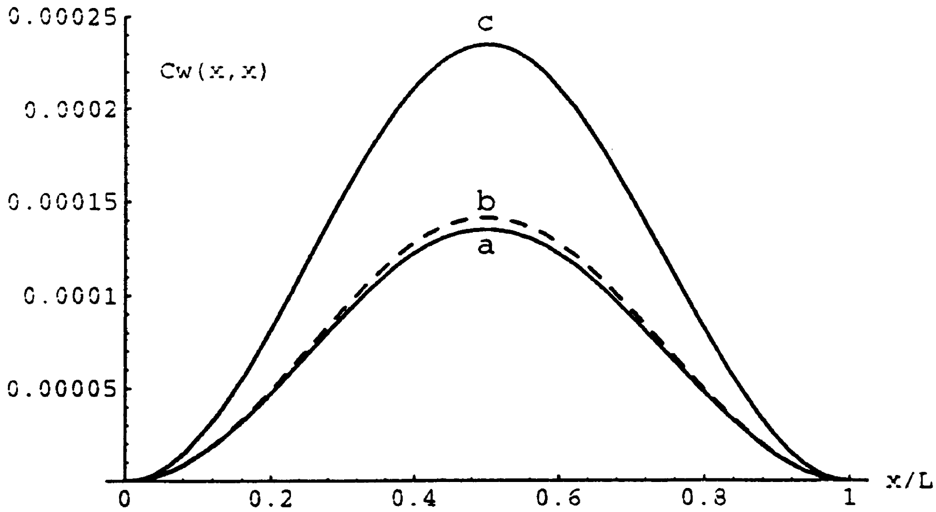

Fig. 6. Comparison of the response variance of the beam, under random loading of unit variance, for various representations of the flexibility: (a) deterministic flexibility; (b) flexibility represented as a random variable; (c) flexibility represented as a random field.

$$\bar{m}(x) = E[\alpha_q]M(x); \quad C_m(x, y) = M(x)M(y)\,\mathrm{Var}\,[\alpha_q] \tag{56}$$

and eqn (52) reduces to eqn (45).

Figure 6 shows a comparison of the response variance of the beam, for various representations of the flexibility.

6. Conclusion

Four different classes of problems for deformation of the Burnoulli–Euler beams, involving stochastic flexibility and load, are considered in this study. New exact solutions are derived for the probabilistic characteristics of the beam's bending moment and displacement. The solutions are obtained by first appropriate differential equations and boundary conditions. The derived exact solutions can serve as benchmark problems: various approximate and numerical solutions, including the finite element method, can be confronted with the derived benchmark solutions.

Appendix: Stochastic clamped-free beam under stochastic loading

If the beam is clamped at $x = 0$ and free at $x = L$ and is subjected to a load $q(x)$, the governing equations for the bending moment are given by eqns (2)–(4). The boundary conditions read

$$m(L) = 0; \quad \frac{\mathrm{d}m(L)}{\mathrm{d}x} = m'(L) = 0 \tag{A1}$$

For an arbitrary bending moment $m(y)$, we obtain

$$C_m(L, y) = E\{[m(L) - \bar{m}(L)]\,[m(y) - \bar{m}(y)]\} = 0$$

$$\frac{\partial C_m(L, y)}{\partial x} = E\{[m'(L) - \bar{m}'(L)]\,[m(y) - \bar{m}(y)]\} = 0 \tag{A2}$$

Similarly

$$C_m(x, L) = 0; \quad \frac{\partial C_m(x, L)}{\mathrm{d}y} = 0 \tag{A3}$$

The boundary conditions (A2), (A3) in explicit form become, after substituting into eqn (8),

$$f_1(L) + yf_2(L) + g_1(y) + Lg_2(y) + \phi(L, y) = 0; \quad f'_1(L) + yf'_2(L) + g_2(y) + \phi_1(L, y) = 0$$

$$f_1(x) + Lf_2(x) + g_1(L) + xg_2(L) + \phi(L, x) = 0; \quad f_2(x) + g'_1(L) + xg'_2(L) + \phi_1(L, x) = 0 \tag{A4}$$

One first determines $f_2(x)$ from the fourth equation; the third equation then yields $f_1(x)$ after substituting $f_2(x)$ into it; the second equation yields $g_2(y)$, whereas the first equation yields $g_1(y)$.

The solution of covariance function becomes

$$C_m(x, y) = \phi(x, y) - \phi(L, x) - \phi(L, y) + (L - y)\phi_1(L, x) + (L - x)\phi_1(L, y) + \phi(L, L)$$
$$+ (L - x)(L - y)\phi_{12}(L, L) - (L - x)\phi_1(L, L) - (L - y)\phi_1(L, L), \quad \text{for } x \geqslant y \tag{A5}$$

$$C_m(x, y) = \phi(y, x) - \phi(L, x) - \phi(L, y) + (L - y)\phi_1(L, x) + (L - x)\phi_1(L, y) + \phi(L, L)$$
$$+ (L - x)(L - y)\phi_{12}(L, L) - (L - x)\phi_1(L, L) - (L - y)\phi_1(L, L), \quad \text{for } x \leqslant y \tag{A6}$$

where

$$\phi_1(u, v) = \frac{\partial \phi(u, v)}{\partial u}; \quad \phi_2(u, v) = \frac{\partial \phi(u, v)}{\partial v}; \quad \phi_{12}(u, v) = \frac{\partial^2 (u, v)}{\partial u\, \partial v} \tag{A7}$$

To obtain boundary conditions for the covariance function we first note that the boundary conditions for displacement for the beam are:

$$w(0) = 0; \quad w'(0) = 0 \tag{A8}$$

For an arbitrary displacement $w(y)$, we have

$$C_w(0, y) = E\{[w(0) - \bar{w}(0)]\,[w(y) - \bar{w}(y)]\} = 0$$

$$\frac{\partial C_w(0, y)}{\partial x} = E\{[w'(0) - \bar{w}'(0)]\,[w(y) - \bar{w}(y)]\} = 0 \tag{A9}$$

Similarly

$$C_w(x,0) = 0; \quad \frac{\partial C_w(x,0)}{\mathrm{d}y} = 0 \tag{A10}$$

In explicit form boundary conditions (A9), (A10) read

$$F_1(0) + yF_2(0) + G_1(y) + \chi(y,0) = 0; \quad F_1(x) + G_1(0) + xG_2(0) + \chi(x,0) = 0$$
$$F'_1(0) + yF'_2(0) + G_2(y) + \chi_2(y,0) = 0; \quad F_2(x) + G'_1(0) + xG'_2(0) + \chi_2(x,0) = 0 \tag{A11}$$

By solving for $F_1(x)$, $F_2(x)$, $G_1(y)$ and $G_2(y)$ the solution of covariance function becomes

$$C_w(x,y) = \chi(x,y) - \chi(x,0) - \chi(y,0) - x\chi_2(y,0) - y\chi_2(x,0) + \chi(0,0) + xy\chi_{12}(0,0) + x\chi_1(0,0) + y\chi_1(0,0), \quad \text{for } x \geqslant y \tag{A12}$$

$$C_w(x,y) = \chi(y,x) - \chi(x,0) - \chi(y,0) - x\chi_2(y,0) - y\chi_2(x,0) + \chi(0,0) + xy\chi_{12}(0,0) + x\chi_1(0,0) + y\chi_1(0,0); \quad \text{for } x \leqslant y \tag{A13}$$

where

$$\chi_1(u,v) = \frac{\partial\chi(u,v)}{\partial u}; \quad \chi_2(u,v) = \frac{\partial\chi(u,v)}{\partial v}; \quad \chi_{12}(u,v) = \frac{\partial^2\chi(u,v)}{\partial u\,\partial v} \tag{A14}$$

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